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LETTER TO THE EDITOR

Self-transform operators

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Abstract
The concept of self-Fourier functions, functions that equal their Fourier transform, is considered using differential operators. The goal is to analyse these functions and determine their properties without evaluating any Fourier, or any other type, transform integrals. Certain known results are generalized and the theory is extended to include integral and fractional-differential operators. Moreover, it is shown that the problem of defining these functions, in its original formulation, is equivalent to this method and in doing so, the concept of a *Fourier eigenoperator* is introduced. We also describe a procedure for applying this approach to general cyclic transforms.

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The importance of Fourier transforms in many areas of physics and mathematics is unquestionable. Indeed, because of its many properties the Fourier transform (FT) pair has been an irreplaceable tool in the theory of boundary value problems (BVPs). We focus in this letter on a special class of functions associated with Fourier transforms, the self-Fourier functions (SFFs); these functions equal their FT. Applications of SFFs can be found in such diverse areas such as optics [1, 2], quantum mechanics [3, 4], analytic number theory [5, 6] and coherent laser design [7–9].

In general, SFFs are special solutions of the eigenvalue problem

$$\mathcal{F}\{f\}(x) = \mu f(x) \tag{1}$$

for $\mu = 1$, where as usual

$$\mathcal{F}\{f\}(x) = \hat{f}(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) \exp(\mathrm{i}\omega x)\, \mathrm{d}\omega.$$

The complete set of eigenvalues is $\mu = \pm 1, \pm \mathrm{i}$, corresponding to self-Fourier ($\mu = 1$), skew-Fourier ($\mu = -1$), i-Fourier ($\mu = \mathrm{i}$) or skew-i-Fourier ($\mu = -\mathrm{i}$) functions, respectively. We shall refer to this set of functions as *Fourier eigenfunctions* (FEFs). The complete set of SFFs is then defined by the generating formula [10, 11]

$$f(x) = g(x) + \hat{g}(x) \tag{2}$$

where $g(x)$ is an arbitrary, even function of x. The main complication here is that given the function $g(x)$ one needs to evaluate its FT, a task often difficult, if even possible. Moreover, even given the function $f(x)$ there exists an infinite family of functions $g(x)$ that will satisfy equation (2), making this decomposition non-unique. Until recently [11], no systematic way of finding these functions existed.

In a recent article [11] new results were presented on the theory of FEFs by considering linear differential operators that commute with the Fourier transform. The analysis was based on the properties of the operator L_{mn}, defined as

$$L_{mn}f \equiv x^m \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \pm \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^n f) = x^m f^{(n)} \pm (x^n f)^{(m)}$$

with (m, n) being an ordered pair of non-negative integers. It was shown that these operators commute with the Fourier operator to a constant; that is,

$$\mathcal{F}\{L_{mn}f\} = \pm(-\mathrm{i})^{m+n} L_{mn}\{\mathcal{F}f\}. \tag{3}$$

If a homogeneous BVP is formed by applying L_{mn} to a function $f(x)$ and imposing appropriate conditions of integrability or decay at infinity, then solutions can be found which satisfy these conditions and are invariant under the Fourier transform. Thus the equations

$$x^m \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \pm \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^n f) = 0$$

will generate SFFs. The functions $g(x)$ are now defined such that

$$g(x) = x^m f^{(n)} \qquad \text{or} \qquad g(x) = (x^n f)^{(m)}.$$

This provides a method to define these generating functions from a given SFF $f(x)$. Provided $m + n = 4k(2k)$, where k is an integer, the sum (difference) of the above terms will generate another SFF from equation (2). The rest of the FEFs corresponding to other Fourier eigenvalues ($\mu \neq 1$) can be generated in an entirely similar way. An example is

$$L_{22}^{+} f \equiv 2x^2 f'' + 4xf' + 2f = 0$$

with $|x|^{-1/2}\cos(\sqrt{3}x/2)$ and $|x|^{-1/2}\sin(\sqrt{3}x/2)$ being the solutions . The cosine solution is even and self-Fourier, while the sine solution is odd and i-Fourier. This generalizes to complex component the previously known SFF $|x|^{-1/2}$, which satisfies $L_{11}^{+} f = 0$.

Now consider the possibility of negative integer values for (m, n). The requirement $f(\pm\infty) = 0$ is replaced by $\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty$, allowing functions that are simply bounded to be included in this formalism. The resulting expressions will reproduce the cases for positive (m, n) values via a one-to-one mapping between the regimes. To see this, assume $(m, n) = (-M, -N)$ for some non-negative integers (M, N). The operator $L_{(-M,-N)}$ generates the equation

$$L_{(-M,-N)} f \equiv x^{-M} f^{(-N)} \pm (x^{-N} f)^{(-M)} = 0 \tag{4}$$

where negative-order differentiation is interpreted as repeated integration. If we define Q to be either term in this expression, we can transform equation (4) into $L_{NM} Q = 0$. For example, substituting $Q \equiv x^{-M} f^{(-N)}$ yields

$$L_{(-M,-N)} f = Q \pm (x^{-N}(x^M Q)^{(N)})^{(-M)} = 0 \Rightarrow$$
$$x^N Q^{(M)} \pm (x^M Q)^{(N)} \equiv L_{NM} Q = 0.$$

So $L_{(-M,-N)}$ maps to L_{NM}. Thus Q and $f = (x^M Q)^{(N)}$ will both be FEF solutions to these equations, or equivalently, both f and $Q = x^{-M} f^{(-N)}$ will be FEFs. Similarly we can start with the positive-index operator L_{mn} and transform it into $L_{(-n,-m)}$ with comparable results. Hence each operator L_{mn} can generate FEFs, and solving $L_{mn} f = 0$ in general is equivalent to solving $L_{(-n,-m)} f = 0$. The indices m and n can also be of opposite sign in this argument, in which case both L_{mn} and $L_{(-n,-m)}$ will generate homogeneous integro-differential equations, which can easily be converted to differential equations via simple differentiation.

The case $m = -n$ is somewhat special since it remains invariant under the Q-substitution. For example, taking $Q = x^{-n} f^{(n)}$, the equation $L_{(-n,n)} f = 0$ becomes $L_{(-n,n)} Q = 0$. Hence f, $Q = (x^n f)^{(-n)}$, or $Q = x^{-n} f^{(n)}$ satisfy the same equation and are all FEF. An example of the general equivalence with negative index is

$$L^{+}_{(0,-2)} f \equiv \int \left(\int f \, \mathrm{d}x \right) \mathrm{d}x + x^{-2} f = 0$$

which is equivalent to $L^{+}_{20} Q = 0$ under $Q = \int \left(\int f \, \mathrm{d}x \right) \mathrm{d}x$ or $Q = x^{-2} f$ this was shown to have SFF solutions in [11]. The equation

$$L^{+}_{(-1,1)} f \equiv x^{-1} \frac{\mathrm{d}f}{\mathrm{d}x} + \int (xf) \, \mathrm{d}x = 0$$

can be used to illustrate the special case $m = -n$. It is straightforward to verify that the known SFF $\exp(\mathrm{i}x^2/2)$ satisfies this equation. The Gaussian $\exp(-x^2/2)$, probably the most common SFF, satisfies $L^{-}_{(-1,1)} f = 0$.

Furthermore, through the definition of fractional-order derivatives, we can extend (m, n) to any real values. The standard product-derivative transformation theorem for the Fourier transform can be used to define derivatives of non-integer order [12], so that $f^{(m)}(x)$ is defined to be the inverse transform of $(-\mathrm{i}x)^m \hat{f}(x)$ for any real m. Thus equation (3) holds for any real-ordered pair (m, n). An alternative definition of the fractional derivative generalizes the Cauchy rule for repeated integration to non-integer orders. Interpretation of the expression $L_{mn} f$ for non-integer values of (m, n) will lead to Volterra-type integral equations. For example, the operator $L_{(0,1/2)}$ yields the semi-differential equation

$$L_{(0,1/2)} f \equiv \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} f \pm x^{1/2} f = 0 \tag{5}$$

which is equivalent to

$$L_{(0,1/2)} f \equiv \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{f(t)}{(x-t)^{1/2}} \, \mathrm{d}t \pm x^{1/2} f = 0 \tag{6}$$

on the positive half-line. Equation (5) can be solved by generalized Frobenius-series methods in conjunction with standard theorems of fractional calculus [13]. With a little further manipulation, equation (6) can be solved by Neumann series methods. The even extension of a suitably decaying or integrable solution will be a SFF on the whole real line. Hereafter and unless otherwise specified, we shall assume that the pair (m, n) are real numbers.

We now proceed to show that the problem as stated in equation (1) and the equations $L_{mn} f = 0$ are equivalent. To do this, we define the Fourier transform $\hat{L}$ of a linear differential operator L to be that operator whose action on the transformed function $\hat{f}$ results from transforming the action of L on f, i.e. $\widehat{Lf} \equiv \hat{L} \hat{f}$. Then a *Fourier eigenoperator* is a linear differential operator which is formally invariant under the Fourier transform; namely, $\hat{L} = cL$ for some constant c, complex in general. With these definitions in mind, it is clear that the operator L_{mn} is an eigenoperator of the Fourier transform:

$$\widehat{L_{mn} f} = \pm(-\mathrm{i})^{m+n} L_{mn} \hat{f} \equiv \hat{L}_{mn} \hat{f} \Rightarrow \hat{L}_{mn} = c L_{mn} \tag{7}$$

where the commuting constant is $c = \pm(-\mathrm{i})^{m+n}$. In terms of operators, this reads $\mathcal{F} L_{mn} = c L_{mn} \mathcal{F}$. The equation $\hat{L} = cL$ can be thought of as defining the eigenoperator problem for the Fourier transform. Any operator satisfying this equation is an eigenoperator of the

Fourier transform by virtue of commuting with the Fourier operator to the constant c, and its nullspace functions and eigenfunctions are candidate eigenfunctions of the Fourier operator. The essential point is that the linear differential operators L_{mn} that commute with $\mathcal{F}$ to a constant are precisely the eigenoperators of $\mathcal{F}$. The operator $L_{02}^{-} \equiv \mathrm{d}^2/\mathrm{d}x^2 - x^2$, from the eigenproblem for the Gauss–Hermite functions, is the most commonly cited example of such an operator.

To further clarify the relationship between the operator L_{mn} and the Fourier eigenproblem equation (1), we form the homogeneous equation $L_{mn} f = 0$. Then $\widehat{L_{mn} f} = cL_{mn}\hat{f} = 0$, so the problem for $\hat{f}$ is formally identical. Applying the same BCs to the problem for $\hat{f}$ yields $\hat{f} = \mu f$, since f and $\hat{f}$ must be proportional. Note that the symmetry of f will determine whether the Fourier eigenvalue μ is ± 1 (even) or $\pm$ i (odd). Similarly, the eigenvalue problem $L_{mn} f = \lambda f$ yields $L_{mn}\hat{f} = (\lambda/c)\hat{f}$. This shows that if f is an eigenfunction of L_{mn} with eigenvalue λ, then $\hat{f}$ is also an eigenfunction of L_{mn} with eigenvalue λ/c. However, unless $c = 1$, we cannot invoke the proportionality argument relating f and $\hat{f}$; the problems must be formally equivalent with the same eigenvalue for this to hold. Furthermore, the eigenvalue must be non-degenerate. Thus for $L_{mn}^{+}(L_{mn}^{-})$, we need $m + n = 4k(4k + 2)$, k is an integer, in order to make the additional claim that the eigenfunction f of L_{mn} is also a FEF.

In addition, the operators L_{mn} give rise to self-adjoint boundary-value problems—in fact, the two terms of $L_{mn} f$ are virtually adjoints of each other to begin with. Since any operator plus its adjoint is self-adjoint, $L_{mn} + L_{mn}^{\dagger}$ is a self-adjoint operator for general (m, n). Regardless of the sign choice in $L_{mn}^{\pm}$ and the values of (m, n), we always have the defining feature $\widehat{L_{mn}} = \pm(-\mathrm{i})^{m+n} L_{mn}$. If we consider such a self-adjoint operator L with $c = 1$, then the operators L and $\mathcal{F}$ commute exactly since the commutator $[L, \mathcal{F}] \equiv L\mathcal{F} - \mathcal{F}L = (1 - c)L\mathcal{F} = 0$. Also, since $\mathcal{F}^{-1} = \mathcal{F}^{\dagger}$, the Fourier operator is unitary. A standard result in operator theory states that a pair of commuting self-adjoint or unitary operators shares a common set of eigenfunctions. Thus L and $\mathcal{F}$ share a common set of eigenfunctions, as we have shown above.

Finally, we propose a general framework for finding the eigenfunctions of a periodic linear transform by finding the eigenoperators of this transform. A transform T is N-periodic or N-cyclic, if its application N times in succession to a function f reproduces that function. Several such transforms with optical applications were briefly considered in [2], [14] and [15] along with their eigenfunctions. The eigenproblem for such a transform gives

$$Tf = \mu f \Rightarrow T^N f = \mu^N f \Rightarrow \mu^N = 1$$

since $T^N \equiv I$, the identity operator. Hence the eigenvalues of T are precisely the Nth roots of unity. Let A_0 be a linear differential operator, and let $A_1, \ldots, A_{N-1}$ be defined by $TA_k = cA_{k+1}$ for some complex constant c, with $0 \leqslant k \leqslant N - 1$; we take $A_N = A_0$. Then $A_k = T^k A_0/c^k$. If we then define

$$L \equiv \sum_{k=0}^{N-1} \gamma_k A_k = \sum_{k=0}^{N-1} \gamma_k \frac{T^k A_0}{c^k}$$

with weighting coefficients γ_k, and consider the action of T on L, we have

$$TL = \sum_{k=0}^{N-1} \gamma_k T A_k = \sum_{k=0}^{N-1} \gamma_k \frac{T^{k+1} A_0}{c^k} = \sum_{k=0}^{N-1} \gamma_k c^{k+1} \frac{A_{k+1}}{c^k} = c \sum_{k=0}^{N-1} \gamma_k A_{k+1}$$

since T is linear. Requiring L and T to commute to the constant c gives the fundamental commutation relation $TL = cLT(\hat{L} = cL)$. This relation holds if $\gamma_0 = \gamma_1 = \cdots = \gamma_{N-1}$.

Since L is linear we may take $\gamma_0 = 1$ without loss of generality. Hence the operators $L = \sum_{k=0}^{N-1} A_k$ all commute with T to their respective constant c. These commuting operators L are precisely the eigenoperators of T. As for the possible values of c, we apply T^{N-1} to both sides of the fundamental commutation relation to obtain

$$TL = cLT \Rightarrow L = cT^{N-1}(LT) = c^N L(T^N) = c^N L \Rightarrow c^N = 1.$$

So the commuting constant c is itself an eigenvalue of the transform T.

We may invoke the discussion above to conclude that the equation $Lf = 0$ will generate solutions which are candidate eigenfunctions of T: $Tf = \mu f$, subject to requirements of transformability under T. As for the eigenproblem in L, we find that if f is an eigenfunction of L with eigenvalue λ, then Tf is also an eigenfunction of L with eigenvalue λ/c. However, unless $c = 1$, we cannot conclude anything further. Only if $c = 1$, and the eigenvalue λ is simple, can we infer that the L-eigenfunction f is also a T-eigenfunction. If we specialize these results to the Fourier transform $T = \mathcal{F}$, then

$$N = 4, \qquad c^4 = 1, \qquad A_0 = x^m \frac{\mathrm{d}^n}{\mathrm{d}x^n}, \qquad A_1 = \frac{(-\mathrm{i})^{m+n}}{c} \frac{\mathrm{d}^m}{\mathrm{d}x^m} x^n,$$
$$A_2 = \left(\frac{(-\mathrm{i})^{m+n}}{c}\right)^2 A_0, \qquad A_3 = \left(\frac{(-\mathrm{i})^{m+n}}{c}\right)^2 A_1.$$

Choosing $c = \pm(-\mathrm{i})^{m+n}$ yields $A_2 = A_0$, $A_3 = A_1$; ignoring duplicated terms, the resulting eigenoperator $L = A_0 + A_1 = x^m f^{(n)} \pm (x^n f)^{(m)}$ is precisely the operator L_{mn} that we have considered in our analysis. On the other hand, choosing $c = \pm\mathrm{i}(-\mathrm{i})^{m+n}$ yields $A_2 = -A_0$, $A_3 = -A_1$ and all terms in L cancel. The results above can easily be specialized to the other transforms considered in [2] and [14].

Finally we note that there is nothing inherently special about the transform eigenvalue $\mu = 1$; the discussion in this letter can be modified in a straightforward way to correspond to eigenfunctions of $\mathcal{F}$ (or the more general T) with whatever eigenvalue μ one desires. Eigenfunctions of T can therefore be determined without evaluating any T-transforms of generating functions.

To summarize, we have generalized the method presented in [11] to analyse and characterize eigenfunctions of the Fourier transform. The use of differential eigenoperators of positive, negative and fractional order allows one to study these functions without having to evaluate any transform integrals. The same approach was finally shown to have useful application in the case of any general linear cyclic transform.

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